

# On some binomial coefficients related to the evaluation of $\tan(nx)$

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## Abstract

The purpose of this paper is to study some binomial coefficients which are related to the evaluation of  $\tan(nx)$ . We present a connection between these binomial coefficients and the coefficients of a family of derivative polynomials for tangent and secant.

**Keywords:** Tangent function; Secant function; Binomial coefficients; Differential operator

## 1 Introduction

Denote by  $D$  the differential operator  $d/dx$ . Throughout this paper, set  $y = \tan(x)$  and  $z = \sec(x)$ . Then  $D(y) = z^2$  and  $D(z) = yz$ . An important tangent identity is given by

$$1 + y^2 = z^2.$$

In 1995, Hoffman [4] considered two sequences of *derivative polynomials* defined respectively by

$$D^n(y) = P_n(y) \quad \text{and} \quad D^n(z) = zQ_n(y)$$

for  $n \geq 0$ . From the chain rule it follows that the polynomials  $P_n(y)$  satisfy  $P_0(y) = y$  and  $P_{n+1}(y) = (1+y^2)P'_n(y)$ , and similarly  $Q_0(y) = 1$  and  $Q_{n+1}(y) = (1+y^2)Q'_n(y) + yQ_n(y)$ . Various refinements of the derivative polynomials have been pursued by several authors (see [3, 5, 6] for instance).

In 1972, Beeler *et al.* found the following elegant identity [1, Item 16]:

$$\tan(n \arctan(t)) = \frac{1}{i} \frac{(1+it)^n - (1-it)^n}{(1+it)^n + (1-it)^n} \quad \text{for } n \geq 0, \quad (1)$$

where  $i = \sqrt{-1}$ . Let

$$R(n, k) = \binom{n}{2k+1} \quad \text{and} \quad T(n, k) = \binom{n}{2k}.$$

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We can now present the following equivalent version of (1):

$$\tan(nx) = \frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k R(n, k) \tan^{2k+1}(x)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k T(n, k) \tan^{2k}(x)},$$

where  $x = \arctan(t)$  (see [7, A034839, A034867] for details). In the sequel to the work of Beeler *et al.*, many other methods have been given to compute  $\tan(nx)$ . Several of them have in common the use of angle addition formula. For example, Szmulowicz [8] obtained a generalized tangent angle addition formula. The purpose of this paper is to explore some further applications of the numbers  $R(n, k)$  and  $T(n, k)$ .

Using the following recurrence relations [2, p. 10]:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{and} \quad \binom{n}{k} = \frac{n-k+1}{k} \binom{n}{k-1},$$

it can be easily verified that

$$nR(n+1, k) = (n+2k+1)R(n, k) + (n-2k+1)R(n, k-1) \quad (2)$$

and

$$nT(n+1, k) = (n+2k)T(n, k) + (n-2k+2)T(n, k-1). \quad (3)$$

For  $n \geq 0$ , we always assume that

$$(Dz)^{n+1}(z) = (Dz)(Dz)^n(z) = D(z(Dz)^n(z))$$

and

$$(Dz)^{n+1}(y) = (Dz)(Dz)^n(y) = D(z(Dz)^n(y)).$$

In this paper we consider the expansions of  $(Dz)^n(z)$  and  $(Dz)^n(y)$ , where the numbers  $R(n, k)$  and  $T(n, k)$  appear in a natural way.

## 2 Polynomials related to $(Dz)^n(z)$ and $(Dz)^n(y)$

For  $n \geq 0$ , we define

$$(Dz)^n(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} M(n, k) y^{n-2k} z^{n+2k+1} \quad (4)$$

and

$$(Dz)^n(y) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} N(n, k) y^{n-2k+1} z^{n+2k}. \quad (5)$$

For example, when  $n = 1$ , we have

$$(Dz)(z) = D(z^2) = 2yz^2 \quad \text{and} \quad (Dz)(y) = D(zy) = y^2z + z^3.$$

**Theorem 1.** For  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , the numbers  $M(n, k)$  satisfy the recurrence relation

$$M(n+1, k) = (n+2k+2)M(n, k) + (n-2k+2)M(n, k-1) \quad (6)$$

with the initial conditions  $M(0, 0) = 1$  and  $M(0, k) = 0$  for  $k \geq 1$ , and the numbers  $N(n, k)$  satisfy the recurrence relation

$$N(n+1, k) = (n+2k+1)N(n, k) + (n-2k+3)N(n, k-1) \quad (7)$$

with the initial conditions  $N(0, 0) = 1$  and  $N(0, k) = 0$  for  $k \geq 1$ .

*Proof.* Note that

$$\begin{aligned} (Dz)^{n+1}(z) &= (Dz)(Dz)^n(z) \\ &= \sum_{k \geq 0} (n+2k+2)M(n, k)y^{n-2k+1}z^{n+2k+2} + \sum_{k \geq 0} (n-2k)M(n, k)y^{n-2k-1}z^{n+2k+4}. \end{aligned}$$

Thus we obtain (6). Similarly, we get (7).  $\square$

Combining (2), (3), (6) and (7), we get the following result.

**Corollary 2.** For  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , we have  $M(n, k) = n!R(n+1, k)$  and  $N(n, k) = n!T(n+1, k)$ .

Recall that  $z^2 = 1 + y^2$ , we define

$$(Dz)^n(z) = \begin{cases} (2m)!zR_{2m+1}(y) & \text{if } n = 2m, \\ (2m+1)!R_{2m+2}(y) & \text{if } n = 2m+1; \end{cases} \quad (8)$$

and

$$(Dz)^n(y) = \begin{cases} (2m)!T_{2m+1}(y) & \text{if } n = 2m, \\ (2m+1)!zT_{2m+2}(y) & \text{if } n = 2m+1, \end{cases}$$

where  $m \geq 0$ . We now present explicit formulas for the polynomials  $R_n(y)$  and  $T_n(y)$ .

**Theorem 3.** For  $n \geq 1$ , we have

$$R_n(y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} R(n, k)y^{n-2k-1}(1+y^2)^{\lfloor \frac{n}{2} \rfloor + k} \quad (9)$$

and

$$T_n(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} T(n, k)y^{n-2k}(1+y^2)^{\lfloor \frac{n-1}{2} \rfloor + k}. \quad (10)$$

*Proof.* We only prove the explicit formula for  $R_{2m+1}(y)$  and the others can be proved in a similar way. Combining (4) and (8), we obtain

$$(Dz)^{2m}(z) = (2m)! \sum_{k \geq 0} R(2m+1, k)y^{2m-2k}z^{2m+2k+1}.$$

Therefore, we get

$$\begin{aligned} R_{2m+1}(y) &= \sum_{k \geq 0} R(2m+1, k) y^{2m-2k} z^{2m+2k} \\ &= \sum_{k \geq 0} R(2m+1, k) y^{2m-2k} (1+y^2)^{m+k}, \end{aligned}$$

and the statement immediately follows.  $\square$

Using (9) and (10), the first few terms of  $R_n(y)$  and  $T_n(y)$  can be calculated directly as follows:

$$R_1(y) = 1, R_2(y) = 2y + 2y^3, R_3(y) = 1 + 5y^2 + 4y^4, R_4(y) = 4y + 16y^3 + 20y^5 + 8y^7;$$

$$T_1(y) = y, T_2(y) = 1 + 2y^2, T_3(y) = 3y + 7y^3 + 4y^5, T_4(y) = 1 + 9y^2 + 16y^4 + 8y^6.$$

For  $n \geq 1$ , we define

$$\tilde{R}_n(y) = \begin{cases} T_{2m}(y) & \text{if } n = 2m, \\ R_{2m+1}(y) & \text{if } n = 2m + 1; \end{cases}$$

and

$$\tilde{T}_n(y) = \begin{cases} R_{2m}(y) & \text{if } n = 2m, \\ T_{2m+1}(y) & \text{if } n = 2m + 1. \end{cases}$$

Let  $\tilde{R}_n(y) = \sum_{k=1}^n \tilde{R}(n, k) y^{2k-2}$  and  $\tilde{T}_n(y) = \sum_{k=1}^n \tilde{T}(n, k) y^{2k-1}$ . For  $1 \leq n \leq 5$ , the coefficients of  $\tilde{R}_n(y)$  can be arranged as follows with  $\tilde{R}(n, k)$  in row  $n$  and column  $k$ :

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 2 & & & & \\ 1 & 5 & 4 & & & \\ 1 & 9 & 16 & 8 & & \\ 1 & 14 & 41 & 44 & 16 & \end{array}$$

For  $1 \leq n \leq 5$ , the coefficients of  $\tilde{T}_n(y)$  can be arranged as follows with  $\tilde{T}(n, k)$  in row  $n$  and column  $k$ :

$$\begin{array}{cccccc} 1 & & & & & \\ 2 & 2 & & & & \\ 3 & 7 & 4 & & & \\ 4 & 16 & 20 & 8 & & \\ 5 & 30 & 61 & 52 & 16 & \end{array}$$

It should be noted that the number  $\tilde{R}(n, k)$  is the number of  $k$ -part order-consecutive partition of the set  $\{1, 2, \dots, n\}$  (see [7, A056242]). The numbers  $\tilde{T}(n, k)$  appear as A210753 in [7].

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